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CAUSTICS AND EVOLUTES FOR CONVEX PLANAR DOMAINS

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Abstract

The caustics for the billiard ball map on an ellipse satisfy a certain evolution property. An operator relating the curvature of a caustic to the curvature of the boundary is defined for the billiard ball map in smooth convex planar domains and is used to derive an equation which characterizes those curves satisfying the evolution property as ellipses.

1. Introduction

Many dynamical systems have been modeled by a billiard ball travelling in a bounded domain in the plane. In considering this problem the objects of interest are periodic orbits—points on the boundary of the region to which a billiard ball returns after a fixed number of reflections—and invariant curves (caustics for optical reflection).

In an elliptic domain, a billiard ball whose trajectory is tangent to an ellipse confocal with the boundary returns to the inside ellipse and its trajectory is again tangential to the inside ellipse, so confocal ellipses define invariant curves and the billiard system on an ellipse is integrable. Birkhoff conjectured that the only integrable convex planar regions with smooth boundaries are ellipses. Seemingly in contrast, it was shown by Moser that in any planar region with a sufficiently smooth boundary, the billiard ball map has enough invariant curves so that the Lebesgue measure of their associated rotation numbers is positive [6] (see also [3]).

What seems to distinguish ellipses from other smooth curves is the *evolution property*—the caustics for an elliptic region are themselves integrable and share their caustics with the elliptic boundary. This property is perhaps easier to see from the point of view of a caustic. If a curve inside a domain is a caustic for the billiard ball map on the domain's boundary, we call the domain's boundary the evolute of the curve. In plane geometry (see Lemma 1), if we loop a string around a closed curve, lean a pen against the string, and draw, then we describe the evolute of the curve. The evolution property

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says that the evolute of an evolute of a curve is also the evolute of the original curve. Said in yet another way, if the caustics for the billiard ball map on B have the evolution property, then when C is a caustic for the billiard ball map on B, any evolute of C is a caustic for the billiard ball map on B.

In order to make use of this evolution property, when the billiard ball map in a region has a caustic we consider the boundary of the region the evolute of the caustic and obtain an operator that relates the curvature of the boundary to the curvature of the caustic. We then use this operator to restate the evolution property in analytic terms and find an equation that must be satisfied by the curvature of curves with the evolution property. This results in our main

Theorem. *The only strictly convex smooth closed planar curve with the property that an evolute of its evolute is an evolute of the original curve is an ellipse.*

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The sections which follow discuss invariant curves and caustics, evolutes and the evolution property, the curvature relating operator, the evolute's curvature, and an equation for the evolution property. The main theorem is proved in the last section.

2. Invariant curves and caustics

Given a strictly convex planar domain Ω, with smooth boundary $\partial\Omega$, the *billiard ball map* on $\partial\Omega$,

$$\beta\colon B^*\partial\Omega \to B^*\partial\Omega,$$

is defined as follows. Let S^*R^2 denote the unit cotangent bundle, and set $B^*\partial\Omega = \{\xi \in T^*\partial\Omega\colon |\xi| \leq 1\}$. We will view $B^*\partial\Omega$ as being embedded in T^*R^2, and S^*R^2 as being embedded in T^*R^2 (we want to be able to use geodesic flow in R^2). Let ϕ_t denote geodesic flow in R^2, and $\pi\colon T^*R^2 \to R^2$ the projection.

There is an inward orientation, $i(p) \in S_pR^2$ (the unit tangent bundle's fiber at p) for $p \in \partial\Omega$. (If $B \subset R^2$ is any set bounding Ω, $\xi \in S_p^*R^2$ satisfies $\xi(i(p)) > 0$, and $\pi(\phi_t\xi) \in \partial B$, then $t > \min\{t > 0 | \pi(\phi_t\xi) \in \partial\Omega\}$.)

Given $\xi \in B_p^*\partial\Omega$, there is a unique inward pointing $w \in S_p^*R^2$ $(w(i(p)) \geq 0)$ with $\xi(v) = w(v)$ for $v \in T_p\partial\Omega \hookrightarrow T_pR^2$.

Define $\rho\colon S_{\partial\Omega}^*R^2 \to B^*\partial\Omega$ by $\rho(w) = \xi$ if $\xi(v) = w(v)$ for all $v \in T_p\partial\Omega$, $p = \pi(w)$. We have just shown that, given an inward pointing orientation of

$\partial\Omega$, ρ has an inverse. If w is not tangent to $\partial\Omega$, then there is a unique $t > 0$ with $\phi_t w \in S^*_{\partial\Omega}R^2$ and

$$\beta(\xi) = \rho\phi_t\rho^{-1}(\xi).$$

If w is tangent to $\partial\Omega$, then we set $\beta(\xi) = \xi$.

Definition 2.1. An *invariant curve* for the billiard ball map on $\partial\Omega$, β, is a smooth section, Ξ, of $B^*\partial\Omega \to \partial\Omega$, such that if $\xi \in \Xi(\partial\Omega)$, then $\beta(\xi) \in \Xi(\partial\Omega)$.

Definition 2.2. A *caustic* for the billiard ball map on $\partial\Omega$ is a simple closed curve $C \subset \Omega$ such that if $\xi \in S^*C$ and $t(\xi)$ is the least $t \geq 0$ with $\pi(\phi_t\xi) \in \partial\Omega$, then there is a $\xi' \in S^*C$ (and a $t(\xi')$, the least $t \geq 0$ with $\pi(\phi_t\xi') \in \partial\Omega$) with $\beta(\rho\phi_{t(\xi)}\xi) = \rho\phi_{t(\xi')}\xi'$.

We may view $\rho^{-1}(\Xi(\partial\Omega)) \subset S^*_{\partial\Omega}R^2$ as a (one-parameter) family of lines (parametrized by any parametrization of $\partial\Omega$). To $\xi \in \rho^{-1}(\Xi(\partial\Omega))$ we associate the line $\{\pi\phi_t\xi | t \in R\}$, where ϕ_t is geodesic flow, and π is projection (see Figure 1).

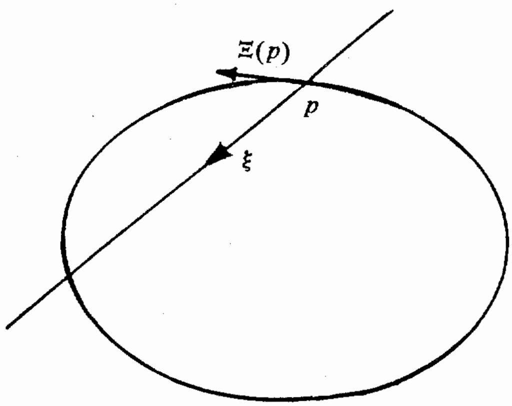

FIGURE 1
The correspondence of Ξ to a family of lines

The role played by the curvature in this setting is clarified if we assume (as we always can near any fixed $s \in \partial\Omega$) that the lines (corresponding to $\Xi(s)$, $s \in \partial\Omega$) are given by $y = y(s) + m(s)x$ in the $x - y$ plane near a fixed line $y = y(s_0) + m(s_0)x$ (we may assume that $m(s) < \infty$ for s near s_0). Then the intersection of $y = y(s) + m(s)x$ and $y = y(s_0) + m(s_0)x$ is given by

$$x = \frac{y(s_0) - y(s)}{m(s) - m(s_0)}.$$

Both numerator and denominator approach zero as s approaches s_0, so this has a unique solution when $k(s_0) = \frac{\partial}{\partial s}m(s_0) \neq 0$ (k is the curvature). So if the curvature is never zero (and hence positive if the curve is closed), there is a well-defined smooth convex caustic corresponding to Ξ. Finally, if Ξ is (C^2) near the boundary—which is itself an invariant curve in $B^*\partial\Omega$—then $k(s) > 0$ for any s.

The construction of the caustic C for a strictly convex Ω shows that the tangent to C at c (when $c = c(\xi)$ this is $\{\pi\phi_t\rho^{-1}(\xi)|t \in R\}$) does not intersect C near c. Since C is closed, this implies that C is convex.

Conversely, it is clear that if C is a smooth strictly convex curve inside Ω then its tangents define a smooth section of $B^*\partial\Omega$. Hence we have shown that given a strictly convex planar domain Ω with a smooth boundary $\partial\Omega$, there is a one-to-one correspondence between invariant curves for the billiard ball map on $\partial\Omega$ in a neighborhood of the boundary and strictly convex smooth caustics for the billiard ball map on $\partial\Omega$ in a neighborhood of $\partial\Omega$. (*Note.* If Ω were not convex and there were a caustic sufficiently close to the boundary, the corresponding "invariant curve" would be discontinuous.)

For completeness we include

Definition 2.3. The billiard ball map on $\partial\Omega$ is *integrable* if there is a neighborhood N of $S^*\partial\Omega$ in $B^*\partial\Omega$ which is included in the images of integral curves for the billiard ball map on $\partial\Omega$, that is, for each $\xi \in N$ there is a $p \in \partial\Omega$ and an integral curve Ξ with $\xi = \Xi(p)$.

3. Evolutes and the evolution property

Let Ω be a strictly convex planar domain with a smooth boundary, and let C be a caustic for the billiard ball map on $\partial\Omega$. We wish to define return points of points in C. Orient C, that is, split S^*C into a disjoint union $S^*C = FS^*C \sqcup BS^*C$. Fix a point $p \in C$ and take $\xi_+(p) \in FS^*_pC$, $\xi_-(p) \in BS^*_pC$ (there is exactly one choice for each of these). Then we define the points $q_+, q_- \in C$ which are, respectively, the *forward* and *backward return points of* p. The point q_+ is such that there are $\xi_-(q_+) \in BS^*_{q_+}C$, $t(\xi_-(q_+)) \geq 0$, and $t(\xi_+(p)) \geq 0$ with

$$\rho(\phi_{t(\xi_-(q_+))}\xi_-(q_+)) = -\rho(\phi_{t(\xi_+(p))}\xi_+(p)) \in B^*\partial\Omega,$$

and $q_- \in C$ is such that there are $\xi_+(q_-) \in FS^*_{q_-}C$, $t(\xi_+(q_-)) \geq 0$, and $t(\xi_-(p)) \geq 0$ with

$$\rho(\phi_{t(\xi_+(q_-))}\xi_+(q_-)) = -\rho(\phi_{t(\xi_-(p))}\xi_-(p)) \in B^*\partial\Omega.$$

Definition 3.1. For a, $b \in C$ denote by $|a \frown b|$ the length of the arc segment of C between a and b. With this and the notation above, we set

$$FQ(p, C, \partial\Omega) = t(\xi_+(p)) + t(\xi_-(q_+)) - |q_+ \frown p|,$$

which we call *the forward reflected distance of C from $\partial\Omega$ at p*, and

$$BQ(p, C, \partial\Omega) = |p \frown q_-| - t(\xi_+(q_-)) - t(\xi_-(p)),$$

called *the backward reflected distance of C from $\partial\Omega$ at p (see Figure 2)*.

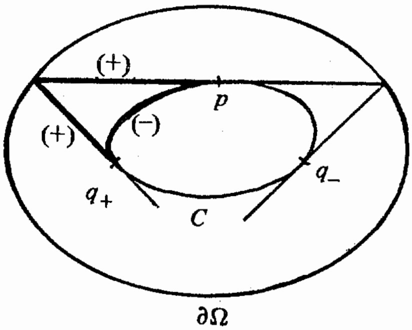

FIGURE 2
The reflected distance of C from ∂C at p

Note that we could define the "reflected distance" for any curve C inside Ω (not necessarily a caustic) by replacing ρ and $-\rho$ above by π, and $B^*\partial\Omega$ by $\partial\Omega$, but this seems to be special to two dimensions.

Lemma 1. *If C is a caustic for the billiard ball map on $\partial\Omega$ (Ω a strictly convex planar domain with a smooth boundary) then the forward reflected distance, $FQ(p, C, \partial\Omega)$, is independent of the point p, as is the backward reflected distance.*

Proof. Consider $FQ(p, c, \partial\Omega)$. Fix $p, p' \in C$, and let $q, q' \in C$ be their forward return points. Let $f \subset C$ be the arc segment between p and p', and let $b \subset C$ be the arc segment between q and q'. Recall our orientation of C, $S^*C = FS^*C \sqcup BS^*C$, and for $\xi \in S^*C$ let $t(\xi)$ denote the first $t \geq 0$ with $\phi_t\xi \in S^*_{\partial\Omega}R^2$. Consider the forward and backward flowout submanifolds F, $B \subset S^*R^2$ given by

$$F = \{\phi_t\xi | \xi \in FS^*C,\ 0 \leq t \leq t(\xi),\ \pi(\xi) \in f\},$$

and

$$B = \{\phi_t\xi | \xi \in BS^*C,\ 0 \leq t \leq t(\xi),\ \pi(\xi) \in b\}.$$

We define a map $\tau\colon F, B \to B^*\partial\Omega$. If $\xi \in F$ or $\xi \in B$, then $\xi = \phi_{t_*}\xi_*$ for some $t_* \geq 0$ and $\xi_* \in S^*C$, and we set $\tau(\xi) = \rho\phi_{t(\xi_*)-t_*}\xi$ (this is geodesic translation to $\partial\Omega$ followed by projection to $B^*\partial\Omega$).

The forward and backward flowouts, F and B above, are Lagrangian with respect to the pullback $\tau^*\omega$ of the cannonical two-form on $B^*\partial\Omega$ because

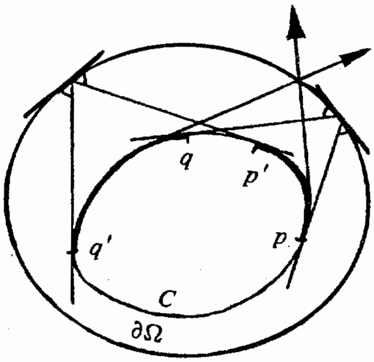

FIGURE 3
The forward and backward flowouts

f and b are curves and $\tau^*\omega$ is invariant under geodesic flow. Using Stoke's theorem we obtain

$$0 = \int_F \omega = \int_{\xi \in \partial F} \iota(\xi)\omega \quad \text{and} \quad 0 = \int_B \omega = \int_{\xi \in \partial B} \iota(\xi)\omega.$$

To conclude the lemma for the forward reflected distance observe that

$$\int_{\partial F} \iota(\xi)\omega + \int_{\partial B} \iota(\xi)\omega = FQ(p, C, \partial\Omega) - FQ(p', C, \partial\Omega)$$

$$(*) \qquad + \int_{P \in A} |\rho(\xi_+(P))| - |\rho(\xi_-(P))|,$$

where A denotes the arc segment $\pi(B) \cap \partial\Omega = \pi(F) \cap \partial\Omega$, $\xi_+(P) \in F_P$ and $\xi_-(P) \in B_P$. When C is a caustic, $|\rho(\xi_+(P))| = |\rho(\xi_-(P))|$, so the integral on the right side of $(*)$ is zero (as are those on the left) and $FQ(p, C, \partial\Omega) = FQ(p', C, \partial\Omega)$.

For the backward reflected distance the proof is the same with the signs and roles of p and q (and of p' and q') reversed.

In fact, if $C \subset \Omega$ is a smooth curve and the reflected distance of C from $\partial\Omega$ is constant, it follows from $(*)$ that C is a caustic for the billiard ball map on $\partial\Omega$.

Let Ω be a strictly convex planar domain with a smooth boundary for which the billiard ball map is integrable. In light of the lemma there is a relation between $\partial\Omega$ and each of the caustics for the billiard ball map on $\partial\Omega$, which invites the following

Definition 3.2. For $Q \in \mathbf{R}$ and $C \subset \Omega$ a caustic for the billiard ball map on $\partial\Omega$, we say that $\partial\Omega$ is *the Q-evolute of C* if $Q \geq 0$ and $FQ(p, C, \partial\Omega) = Q$ for $p \in C$ or if $Q < 0$ and $BQ(p, C, \partial\Omega) = Q$ for $p \in C$.

There are cases in which the caustics for the billiard ball map have an additional property: For the circle, parametrized by arclength s, and with

$(s,\theta) \in S^*_{S^1}R^2$, the billiard ball map can be represented as $\beta(s,\theta) = (s+2\theta, \theta)$. Hence, caustics for the billiard ball map on the circle are concentric circles, so the billiard ball map on any given caustic is itself integrable and the caustics for the billiard ball map on a caustic are also caustics for the billiard ball map on the boundary.

Definition 3.3. We say that the caustics for the billiard ball map satisfy the *evolution property* if the evolute of the evolute of a given caustic is also the evolute of that same caustic.

The caustics for the billiard ball map in an elliptic region which are near the region's boundary also satisfy the evolution property and, indeed, if ellipses are the only integrable planar curves (as conjectured by G. D. Birkhoff), then so do the caustics of all integrable planar domains.

4. The curvature relating operator

We identify smooth strictly convex curves in R^2 by their curvature (see, for example, [4]). That is, to each closed curve we associate its curvature when the curve is parametrized by tangent angle, and to each positive $k \in C^\infty(R/2\pi Z; R)$, with

$$\int_0^{2\pi} \cos(t)\frac{dt}{k(t)} = \int_0^{2\pi} \sin(t)\frac{dt}{k(t)} = 0,$$

we associate the curve with coordinates

$$x(\theta) = \int_0^{\theta} \cos(t)\frac{dt}{k(t)}, \qquad y(\theta) = \int_0^{\theta} \sin(t)\frac{dt}{k(t)}.$$

In this setting we have

Definition 4.1. The *curvature relating operator* is

$$L\colon \mathbf{R} \times C^\infty(S^1;\mathbf{R}) \to C^\infty(S^1;\mathbf{R}),$$

which takes (Q,k) to v, the curvature of the Q-evolute of the curve whose curvature is k.

Proposition 2. *In the setting of Definition* 4.1, $L(-Q,k) = L(Q,k)$.

Proof. If we change the orientation of the curve C with curvature k, we do not change the curvature of the evolute. But $FQ(p,C,\partial\Omega) = -BQ(q,\overline{C},\partial\Omega)$, where q is the forward return point of p, and $\overline{C}$ is C with the reversed orientation.

We would like to examine the curvature relating operator more closely. Let a be a simple closed strictly convex smooth planar curve given by its tangent angle ($0 \leq \theta \leq 2\pi$) and its curvature ($k(\theta) > 0$):

$$a(\theta) = \left(\int_0^\theta \cos(t) \frac{dt}{k(t)}, \quad y_a^0 + \int_0^\theta \sin(t) \frac{dt}{k(t)} \right).$$

Let b be the Q-evolute of a ($Q > 0$), given by its tangent angle ϕ and curvature v. Then for some θ_1, t_1, θ_2, and t_2

$$b(\phi) = a(\theta_1) + t_1(\cos\theta_1, \sin\theta_1) = a(\theta_2) - t_2(\cos\theta_2, \sin\theta_2),$$

and

$$Q = t_1 + t_2 - (s(\theta_2) - s(\theta_1)),$$

where s is the arclength along a, $s(\theta) = \int_0^\theta k^{-1}(t)\,dt$. Since a is assumed to be an invariant curve for the billiard ball map on b, $\phi = (\theta_1 + \theta_2)/2$. As well,

$$t_1 \cos(\theta_1) + t_2 \cos(\theta_2) = \int_{\theta_1}^{\theta_2} \cos(t) k^{-1}(t)\,dt,$$

$$t_1 \sin(\theta_1) + t_2 \sin(\theta_2) = \int_{\theta_1}^{\theta_2} \sin(t) k^{-1}(t)\,dt,$$

$$\begin{aligned} t_1 + t_2 &= \frac{1}{\sin(\theta_2 - \theta_1)} \int_{\theta_1}^{\theta_2} [\sin\theta_2 \cos t - \cos\theta_2 \sin t \\ &\qquad + \cos\theta_1 \sin t - \sin\theta_2 \cos t] k^{-1}(t)\,dt \\ &= \frac{1}{\cos\Delta} \int_{\phi-\Delta}^{\phi+\Delta} \cos(\phi - t) k^{-1}(t)\,dt, \end{aligned}$$

where $\Delta = \theta_2 - \phi = \phi - \theta_1$. Thus

$$Q = \frac{1}{\cos\Delta} \int_{-\Delta}^{\Delta} \cos(u) k^{-1}(\phi + u)\,du - \int_{-\Delta}^{\Delta} k^{-1}(\phi + u)\,du.$$

The right side of the equation above is clearly smooth in Δ near zero, and it turns out to vanish to third order in Δ as we will see in the next section where we continue the computation. In fact, $Q = \Delta^3 f(\Delta)$ where f is smooth in Δ near zero and $f(0) = (12k)^{-1}$. With this observation we can show

Proposition 3. *$L(Q, k)$ is a differential operator (in k) which is smooth in $Q^{2/3}$ for sufficiently small Q.*

Proof. As noted above, $Q = \Delta^3 f(\Delta)$, and since $f(0) = (12k(\theta))^{-1} > 0$ on the entire (compact) curve C, by the implicit function theorem, Δ is smooth in $Q^{1/3}$.

The curvature relating operator L is smooth in Δ because b is smooth in t_1 (and in θ), and

$$\begin{aligned} t_1 &= \frac{1}{\sin 2\Delta} \int_{\phi-\Delta}^{\phi+\Delta} \sin(\phi + \Delta - t) k^{-1}(t)\,dt \\ &= \frac{1}{2\sin\Delta} \int_{-\Delta}^{\Delta} \sin(u) k^{-1}(\phi + u)\,du + \frac{1}{\cos\Delta} \int_{-\Delta}^{\Delta} \cos(u) k^{-1}(\phi + u)\,du, \end{aligned}$$

while

$$\int_{-\Delta}^{\Delta} \sin(u)k^{-1}(\phi+u)\,du \sim \sin\Delta[k^{-1}(\phi+\Delta)+k^{-1}(\phi-\Delta)] + O(\Delta^2),$$

so t_1 is smooth in Δ (Δ near 0). Hence the curvature relating operator is smooth in $Q^{1/3}$.

However, $L(Q,k) = L(-Q,k)$ as shown in Proposition 2, and Q and $Q^{1/3}$ have the same sign, proving that L is smooth in $Q^{2/3}$.

5. The evolute's curvature

For a curve whose evolutes satisfy the evolution property, the evolute of an evolute is an evolute of the original curve. In terms of the curvature relating operator, for a curvature function, k, corresponding to a caustic which is in a collection of caustics satisfying the evolution property and for any (sufficiently small) Q and P, there is an R with

$$L(Q, L(P,k))(\psi) = L(R,k)(\psi) \quad \forall\psi \in [0, 2\pi]. \tag{5.1}$$

Differentiating this equation (perhaps several times) and setting $Q = P = 0$ gives equations that must be satisfied by the curvature function k. We calculate the first nontrivial equation derived from this procedure by means of formal Taylor series in P, Q, and R, and show that it is satisfied only by the curvature functions of ellipses.

Recall that L is smooth in $Q^{2/3}$, so formally

$$Lk \sim \sum_{j=0}^{\infty} L_j k Q^{2j/3}.$$

We also know that $L(0,k) = k$ so L_0 is the identity.

Fix $P, Q > 0$, set $v = L(P,k)$ and $w = L(Q,v)$, and assume that $w = L(R,k)$ for some $R > 0$. Since $k, v, w > 0$, $R^{2/3}$ is smooth in $Q^{2/3}$ and in $P^{2/3}$. Also, when $P = 0$, $R = Q$ and when $Q = 0$, $R = P$, so

$$R^{2/3} = P^{2/3} + Q^{2/3} + P^{2/3}Q^{2/3}G(k, P^{2/3}Q^{2/3}),$$

with G depending on the curvature k (but not on the tangent angle) and jointly smooth in $P^{2/3}$ and $Q^{2/3}$.

For (5.1) to hold, we must have $L(Q, L(P,k)) \sim L(R,k)$ or

$$\begin{aligned}
L_0k + L_1k(P^{2/3} + Q^{2/3}) &+ L_1^{two}kP^{2/3}Q^{2/3} + L_2k(P^{4/3} + Q^{4/3}) \\
&= L_0k + L_1kR^{2/3} + L_2kR^{4/3} + O_3 \\
&= L_0k + L_1k(P^{2/3} + Q^{2/3}) + L_1kP^{2/3}Q^{2/3}G(k,0,0) \\
&\quad + L_2k(P^{4/3} + 2P^{2/3}Q^{2/3} + Q^{4/3}) + O_3,
\end{aligned}$$

where O_3 represents $O(P^{2/3}, Q^{2/3})^3$ and $L_1^{two}k$ denotes the coefficient of $P^{2/3}$ in $L_1(k + L_1 k P^{2/3})$. The equation of the evolution property is thus

$$L_1^{two}k = L_1 k G(k, 0, 0) + 2L_2 k. \tag{5.2}$$

To find L_1 and L_2 we continue the calculation from the preceding section ($b(\phi)$ with the curvature function v is the Q-evolute of $a(\theta)$ whose curvature is k). Recall that

$$Q = \frac{1}{\cos\Delta}\int_{-\Delta}^{\Delta}\cos(u)k^{-1}(\phi+u)\,du - \int_{-\Delta}^{\Delta}k^{-1}(\phi+u)\,du.$$

Let $g(\Delta) = k^{-1}(\phi+\Delta) + k^{-1}(\phi-\Delta)$, and note that $g(-\Delta) = g(\Delta)$. We have

$$\int_{-\Delta}^{\Delta}\cos(u)k^{-1}(\phi+u)\,du = g(0)\Delta + \frac{1}{3!}(d^2g - g)\Delta^3 + \frac{1}{5!}(d^4g - 6d^2g + g)\Delta^5 + O(\Delta^7),$$

$$\int_{-\Delta}^{\Delta}k^{-1}(\phi+u)\,du = g(0) + \frac{1}{3!}d^2g(0)\Delta^3 + \frac{1}{5!}d^4g(0)\Delta^5 + O_7,$$

$$g(0) = 2k^{-1}(\phi), \qquad \frac{d^2g}{d\Delta^2}(0) = 4k^{-3}(k')^2 - 2k^{-2}k'',$$

$$\begin{aligned} Q &= \frac{1}{3!}(-g)\Delta^3 + \frac{1}{5!}(6d^2g + g)\Delta^5 + O_7 \\ &\quad + \left(\frac{1}{2}\Delta^2 + \frac{5}{24}\Delta^4 + O_6\right)\left(g\Delta + \frac{1}{3!}(d^2g - g)\Delta^3 + O_5\right) \\ &= \frac{2}{3}k^{-1}(\phi)\Delta^3 + \left(\frac{2}{15}k^{-3}(k')^2 - \frac{1}{15}k^{-2}k'' + \frac{4}{15}k^{-1}\right)\Delta^5 + O_7, \end{aligned}$$

where we have used $(\cos x)^{-1} = 1 + x^2/2 + x^4/24 + O(x^6)$.

With $q = Q^{1/3}$, we solve for Δ, obtaining

$$\Delta = (\tfrac{3}{2})^{1/3}k^{1/3}q + (\tfrac{1}{20}k'' - \tfrac{1}{10}k^{-1}(k')^2 - \tfrac{1}{5}k)q^3 + O(q^5),$$

where k and its derivatives are evaluated at ϕ.

With $b(\phi) = (x(\phi), y(\phi))$, we have

$$v(\phi) = \left(\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right)^{-1/2},$$

$$d_\phi x = \frac{\cos(\phi-\Delta)}{k(\phi-\Delta)}d_\phi(\phi-\Delta) - t_1\sin(\phi-\Delta)d_\phi(\phi-\Delta) + \cos(\phi-\Delta)d_\phi t_1,$$

$$d_\phi y = \frac{\sin(\phi-\Delta)}{k(\phi-\Delta)}d_\phi(\phi-\Delta) + t_1\cos(\phi-\Delta)d_\phi(\phi-\Delta) + \sin(\phi-\Delta)d_\phi t_1,$$

$$(d_\phi x)^2 + (d_\phi y)^2 = (k^{-1}(\phi - \Delta)(1 - d_\phi \Delta) + d_\phi t_1)^2 + t_1^2(1 - d_\phi \Delta)^2,$$

where t_1 is as in the preceding section (see Proposition 3). Setting $h(\Delta) = k^{-1}(\phi + \Delta) - k^{-1}(\phi - \Delta)$, observe that $h(-\Delta) = -h(\Delta)$,

$$\int_{-\Delta}^{\Delta} \sin(u) k^{-1}(\phi + u)\, du = \tfrac{1}{3} d_\Delta h(0)\Delta^3 + (\tfrac{1}{30} d_\Delta^3 h - \tfrac{1}{30} d_\Delta h)\Delta^5 + O_7,$$

$d_\Delta h(0) = -2k^{-2}k'(\phi)$, and $d_\Delta^3 h(0) = -2k^{-2}k^{(3)}(\phi) + 12k^{-3}k'k'' - 12k^{-4}(k')^3$.

Using the previous calculation for the portion involving cosine,

$$t_1 = k^{-1}(\phi)\Delta + \tfrac{1}{3}k^{-2}k'\Delta^2 + [\tfrac{1}{6}k^{-2}k'' + \tfrac{1}{3}k^{-3}(k')^2 + \tfrac{1}{3}k^{-1}]\Delta^3$$
$$+ [\tfrac{1}{30}k^{-2}k^{(3)} - \tfrac{1}{5}k^{-3}k'k'' + \tfrac{1}{5}k^{-4}(k')^3 + \tfrac{1}{45}k^{-2}k']\Delta^4 + O_5.$$

Now,

$$d_\phi \Delta = \tfrac{1}{3}(\tfrac{3}{2})^{1/3} k^{-2/3}k' q + [\tfrac{1}{20}k^{(3)} + \tfrac{1}{10}k^{-2}(k')^3$$
$$- \tfrac{1}{5}k^{-1}k'k'' - \tfrac{1}{5}k']q^3 + O(q^5),$$

$$k^{-1}(\phi - \Delta) = k^{-1}(\phi) + k^{-2}k'\Delta + \tfrac{1}{2}[-k^{-2}k'' + 2k^{-3}(k')^2]\Delta^2$$
$$+ \tfrac{1}{6}[k^{-2}k^{(3)} - 6k^{-3}k'k'' + 6k^{-4}(k')^3]\Delta^3$$
$$+ [-k^{-2}k^{(4)} + 8k^{-3}k'k^{(3)} + 6k^{-3}(k'')^2$$
$$- 36k^{-4}(k')^2k'' + 24k^{-5}(k')^4]\Delta^4 + O_5,$$

and

$$k^{-1}(\phi - \Delta)(1 - d_\phi \Delta) + d_\phi t_1$$
$$= k^{-1}(\phi) + [-\tfrac{1}{6}k^{-2}k'' + \tfrac{1}{3}k^{-3}(k')^2]\Delta^2 - \tfrac{1}{3}k^{-2}k'\Delta^3$$
$$+ [-\tfrac{1}{120}k^{-2}k^{(4)} + \tfrac{1}{15}k^{-3}k'k^{(3)} + \tfrac{1}{20}k^{-3}(k'')^2 - \tfrac{3}{10}k^{-4}(k')^2k''$$
$$+ \tfrac{1}{45}k^{-2}k'' + \tfrac{1}{5}k^{-5}(k')^4 - \tfrac{2}{45}k^{-3}(k')^2]\Delta^4$$
$$- \tfrac{1}{3}k^{-2}k'\Delta d_\phi \Delta + k^{-1}\Delta^2 d_\phi \Delta$$
$$+ [-\tfrac{1}{30}k^{-2}k^{(3)} + \tfrac{1}{5}k^{-3}k'k'' - \tfrac{1}{5}k^{-4}(k')^3 + \tfrac{4}{45}k^{-2}k']\Delta^3 d_\phi \Delta + O_5$$
$$= k^{-1}(\phi) + (\tfrac{3}{2})^{2/3}[-\tfrac{1}{6}k^{-4/3}k'' + \tfrac{2}{9}k^{-7/3}(k')^2]q^2$$
$$+ (\tfrac{3}{2})^{2/3}[-\tfrac{1}{80}k^{-2/3}k^{(4)} + \tfrac{1}{15}k^{-5/3}k'k^{(3)}$$
$$+ \tfrac{7}{120}k^{-5/3}(k'')^2 - \tfrac{2}{9}k^{-8/3}(k')^2k'' + \tfrac{1}{10}k^{-2/3}k''$$
$$+ \tfrac{1}{9}k^{-11/3}(k')^4 - \tfrac{1}{15}k^{-5/3}(k')^2]q^4$$
$$+ O(q^5).$$

Also,

$$t_1^2(1 - d_\phi \Delta)^2 = (\tfrac{3}{2})^{2/3}k^{-4/3}q^2$$
$$+ (\tfrac{3}{2})^{1/3}[-\tfrac{2}{5}k^{-5/3}k'' + \tfrac{7}{15}k^{-8/3}(k')^2 + \tfrac{3}{5}k^{-2/3}]q^4 + O_5,$$

$$\begin{aligned}(k^{-1}(\phi-\Delta)(1-d_\phi\Delta)+d_\phi t_1)^2 \\ = [k^{-2}(\phi) + (\tfrac{3}{2})^{2/3}(-\tfrac{1}{3}k^{-7/3}k'' + \tfrac{4}{9}k^{-10/3}(k')^2 + k^{-4/3}]q^2 \\ + (\tfrac{3}{2})^{1/3}[-\tfrac{1}{40}k^{-5/3}k^{(4)} + \tfrac{2}{15}k^{-8/3}k'k^{(3)} + \tfrac{19}{120}k^{-8/3}(k'')^2 \\ - \tfrac{5}{9}k^{-11/3}(k')^2k'' - \tfrac{1}{5}k^{-5/3}k'' + \tfrac{8}{27}k^{-14/3}(k')^4 \\ + \tfrac{1}{3}k^{-8/3}(k')^2 + \tfrac{3}{5}k^{-2/3}]q^4 + O_5,\end{aligned}$$

and finally,

$$\begin{aligned}v(\phi) = k(\phi) + (\tfrac{3}{2})^{2/3}[\tfrac{1}{6}k^{2/3}k'' - \tfrac{2}{9}k^{-1/3}(k')^2 - \tfrac{1}{2}k^{5/3}]q^2 \\ + (\tfrac{3}{2})^{1/3}[\tfrac{1}{80}k^{4/3}k^{(4)} - \tfrac{1}{15}k^{1/3}k'k^{(3)} - \tfrac{1}{60}k^{1/3}(k'')^2 \\ + \tfrac{1}{9}k^{-2/3}(k')^2k'' - \tfrac{11}{40}k^{4/3}k'' - \tfrac{1}{27}k^{-5/3}(k')^4 \\ + \tfrac{1}{3}k^{1/3}(k')^2 + \tfrac{21}{80}k^{7/3}]q^4 + O(q^5).\end{aligned} \tag{5.3}$$

In fact, we know that v is even in q, so that this equation holds to sixth order. Equation (5.3) gives L_0k, L_1k, and L_2k as the coefficients of 1, q^2, and q^4 (respectively).

6. An equation for the evolution property

We now return to the setting of three curves with curvatures k, v, and w as in the beginning of the previous section, and find the equation satisfied by the curvature k (the curvature of the curve whose evolutes are assumed to have the evolution property). Using (5.3), we find that

$$\begin{aligned}L_1v = (\tfrac{3}{2})^{2/3}[\tfrac{1}{6}k^{2/3}k'' - \tfrac{2}{9}k^{-1/3}(k')^2 - \tfrac{1}{2}k^{5/3}] \\ + (\tfrac{3}{2})^{1/3}[\tfrac{1}{24}k^{4/3}k^{(4)} - \tfrac{1}{6}k^{1/3}k'k^{(3)} - \tfrac{1}{18}k^{1/3}(k'')^2 + \tfrac{31}{108}k^{-2/3}(k')^2k'' \\ - \tfrac{1}{2}k^{4/3}k'' - \tfrac{8}{81}k^{-5/3}(k')^4 + \tfrac{23}{36}k^{1/3}(k')^2 + \tfrac{5}{8}k^{7/3}]P^{2/3} \\ + O(P^{4/3}),\end{aligned}$$

and (5.2) gives

$$L_1^{two}k - 2L_2k = G(k)L_1k,$$

which becomes

$$\begin{aligned}(\tfrac{3}{2})^{1/3}[\tfrac{1}{60}k^{4/3}k^{(4)} - \tfrac{1}{30}k^{1/3}k'k^{(3)} - \tfrac{1}{45}k^{1/3}(k'')^2 + \tfrac{7}{108}k^{-2/3}(k')^2k'' \\ + \tfrac{1}{20}k^{4/3}k'' - \tfrac{2}{81}k^{-5/3}(k')^4 - \tfrac{1}{36}k^{1/3}(k')^2 + \tfrac{1}{10}k^{7/3}] \\ = G(k)(\tfrac{3}{2})^{2/3}(\tfrac{1}{6}k^{2/3}k'' - \tfrac{2}{9}k^{-1/3}(k')^2 - \tfrac{1}{2}k^{5/3}).\end{aligned} \tag{6.1}$$

Equation (6.1) is the desired evolution property equation, and is satisfied by the curvature of any ellipse and its rotations. For the ellipse with $x^2/a^2 + y^2/b^2 = 1$,

$$k(\theta) = (ab)^{-2}(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}, \quad G = -\tfrac{1}{10}(\tfrac{3}{2})^{-1/3}(ab)^{-4/3}(a^2+b^2).$$

A closer look at this equation shows that

Theorem 4. *The only smooth strictly-convex closed planar curves satisfying the evolution property (Definition 3.3 and equation (5.1)) are ellipses.*

Proof. We need to make some observations about the evolution property equation that will allow us to conclude that the only positive periodic solutions are those whose initial conditions agree with those of the curvatures of ellipses.

First note that if we set $y = -\theta$ and consider the curvature as a function of y the equation remains the same (we are describing the same curve). Next note that since we are only interested in periodic solutions our solution must have a minimum which we may assume to be at $\theta = 0$ by rotation. Returning to the equation and using these observations it follows that $k^{(3)}(0) = k'(0) = 0$.

The evolution property equation is also homogeneous—if k satisfies the equation so does ck for any constant c, and $G(ck) = c^{2/3}G(k)$. Thus we may assume that $G(k) = -1$.

Finally, notice that we can find an ellipse with curvature E such that $G(E) = -1$ and $E''(0) = k''(0)$ (at a minimum $k''(0) > 0$).

We investigate $k(0)$. If we can choose $k(0)$ arbitrarily, then there is a sequence of solutions, k_r, whose first three derivatives (at 0) agree with those of some (fixed) ellipse but so that $k_r(0) \to 0$ as $r \to 0$. But then the fact that the terms in the equation all remain bounded implies that $k' \sim k^{5/12}$ which contradicts the original equation. (Looking at this geometrically, because the equation remains unchanged with $u = -k$, if k is allowed to approach 0 then the corresponding curve must degenerate to a line.)

It is my hope that this result will be helpful in understanding the spectral properties of ellipses, and that integrable curves can be shown to satisfy the evolution property thereby proving that the only integrable smooth strictly convex curves in the plane are ellipses.

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